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Neighboring Optimal Trajectories for Aeroassisted Orbital Transfer Under Uncertainties

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A neighboring optimal guidance scheme is presented for a nonlinear dynamic system in the presence of modeling and measurement uncertainties as applicable to fuel optimal control of an aeroassisted orbital transfer vehicle (AOTV). Here the vehicle is expected to follow a desired reference path with minimum error. The measurements considered include tracking of altitude and altitude rate. The solution approach is based on nonlinear control optimization, deterministic perturbation, stochastic state estimation, and linearized stochastic control that leads to a closed-loop control law. The reference state trajectory and control trajectory are precomputed by solving a two-point boundary-value control problem. Numerical results show that the actual (true) state trajectory is kept near its ideal (reference) state trajectory at the expense of increasing the total cost.

Nomenclature

a_c	= R_c/R_a
a_d	= R_d/R_a
b	= R_a/H_a
C_D	= drag coefficient
C_{D0}	= zero-lift drag coefficient
C_L	= lift coefficient
C_{LR}	= lift coefficient for maximum lift-to-drag ratio
$C_0(\tau)$	= evaluation of $C(\tau)$ along optimal path
c	= normalized lift coefficient, C_L/C_{LR}
D	= drag force
g	= gravitational acceleration
H	= altitude
HEO	= high Earth orbit
h	= normalized altitude, H/H_a
$h_o(\tau)$	= evaluation of $h(\tau)$ along optimal path
J	= performance index
$\bar{J}$	= augmented performance index
K	= induced drag factor
K_f	= filter gain
L	= lift force
LEO	= low Earth orbit
M	= gravitational constant of Earth
m	= vehicle mass
n_{ai}	= measurement noise
OTV	= orbital transfer vehicle
P	= error covariance matrix
P_h, P_v, P_γ	= adjoint variables
$\bar{P}_0$	= covariance of initial state
R_a	= radius of atmosphere boundary
R_c	= radius of low Earth orbit
R_d	= radius of high Earth orbit
r_E	= radius of Earth
r	= radial distance from Earth's center
S	= aerodynamic reference area
t	= time
$u(\tau)$	= general control vector
V	= velocity
V_e	= entry velocity
V_f	= exit velocity
W	= controllability Grammian
w_{ai}	= plant noise
w_i	= zero-mean unit-variance white-noise process
X	= mean-square matrix of δx
$x(\tau)$	= general state vector
$y(\tau)$	= general signal vector
$y_0(\tau)$	= evaluation of signal vector along optimal path
z_i	= noisy measurements, $z_i = y_i +$ noise
β	= inverse atmospheric scale height
γ	= flight path angle
γ_e	= entry flight path angle
γ_f	= exit flight path angle
$\gamma_0(\tau)$	= evaluation of $\gamma(\tau)$ along optimal path
ΔV	= characteristic velocity
ΔV_c	= characteristic velocity at reorbit
ΔV_d	= characteristic velocity at deorbit
Δv	= normalized characteristic velocity
Δv_c	= normalized characteristic velocity at reorbit
Δv_d	= normalized characteristic velocity at deorbit
$\delta(\cdot)$	= general perturbed vector
δc	= normalized lift perturbation
$\delta\Psi$	= general terminal constraint vector
$\delta\hat{x}(\tau)$	= general state estimate vector
$\delta^2\bar{J}$	= second variation of $\bar{J}$
$\delta^2\bar{J}^0$	= expected value of $\delta^2\bar{J}$
Λ	= control gain
μ	= general Lagrange multiplier vector
$\mathcal{H}$	= Hamiltonian
v	= normalized velocity, $V/\sqrt{\mu}/R_a$
$v_0(\tau)$	= evaluation of $v(\tau)$ along optimal path
ρ	= air density
ρ_n	= normalized air density, ρ/ρ_R
ρ_R	= reference air density
τ	= normalized time, $t/R_a\sqrt{R_a/\mu}$
τ_f	= final normalized time
τ_0	= initial normalized time
Ψ	= general terminal constraint

I. Introduction

IN space transportation systems, the concept of aeroassisted orbital transfer opens new mission opportunities, especially with regard to the establishment of a permanent space station. The space-

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based orbit-transfer vehicle (OTV) is planned as a system for transporting payloads between low Earth orbit (LEO) and high Earth orbit (HEO). The aeroassisted orbital transfer vehicle (AOTV) utilizes atmospheric drag during the atmospheric flight to significantly reduce its velocity and hence exit from the atmosphere to enter into an elliptical orbit. Thus, during the atmospheric pass no rocket fuel for retrobraking is utilized. Since less fuel results in an increased payload, optimization of fuel is an important aspect in orbital transfer missions.[1-3]

The purpose of this paper is to take into consideration modeling approximations,[4] simplifications, and random disturbances and to use the well-known linear quadratic Guassian (LQG) theory to the AOTV control problem. This present work represents a continuation of the original effort reported by the authors.[5-7] In the present case the problem of incomplete knowledge of the state vector is considered, which leads to the state estimators based on information available through noisy sensor measurements. Thus, the Kalman-Bucy filter is used.[8-10]

The importance of this work is easily seen if one is concerned with the behavior of the ideal model in an environment that modifies initial and final conditions, plant parameters, and sensor measurements. That is, it is computationally tedious and expensive to repeat a whole nonlinear optimization procedure for every such environment to obtain the new optimal trajectory. However, if such a situation arises, it is preferable to linearize in the neighborhood of the original optimal trajectory, thus resulting in considerably less computational efforts.[11,12]

The objective of the present paper is to devise a neighboring optimal guidance scheme for a nonlinear system with random inputs and imperfect measurements as applicable to an AOTV. We address the fuel-optimal control problem arising in coplanar orbital transfer from HEO to LEO. The performance index chosen here is the total characteristic velocity, which is equal to the sum of the characteristic velocities for deorbit and for reorbit (or circularization). In Sec. II, the deterministic ideal model of the nonlinear dynamic system is presented that describes the atmospheric maneuver of the space vehicle. The optimization problem is then formulated, leading to a two-point boundary-value problem (TPBVP).[11] The solution of the TPBVP is obtained through the use of a multiple shooting code. In Sec. III, the deterministic linear quadratic optimization problem is formulated and the deterministic neighboring control is given. In Sec. IV, the nonlinear stochastic control problem is presented that is referred to as the true model. The various modeling assumptions that lead to the ideal model of Sec. II are taken into account through the introduction of random inputs. Then the linearized stochastic control problem is formulated, and the specifics of the approach taken to obtain the control and filter gains required for the simulation of the Kalman filter are clearly delineated. The numerical algorithms required to solve for the Kalman estimates are also presented. Finally, in Sec. V the numerical data for simulations is given, and the results of the stochastic neighboring control problem are discussed.

II. Deterministic Ideal Model and Optimal Solution

Differential System

Consider a space vehicle that transfers from HEO at radius R_d to LEO at radius R_c (see Fig. 1).[5] Under the usual assumptions,[4] the unnormalized equations of motion for a coplanar atmospheric flight are given in Refs. 5–7.

Using the dimensionless variables and parameters

$$h = \frac{H}{H_a}, \qquad v = \frac{V}{\sqrt{\mu/R_a}}, \qquad \tau = \frac{t}{R_a\sqrt{R_a/\mu}}$$

$$b = \frac{R_a}{H_a}, \qquad \rho_n = \frac{\rho}{\rho_R}, \qquad c = \frac{C_L}{C_{LR}}$$

$$C_{LR} = \sqrt{C_{D0}/K}, \qquad A_1 = \frac{C_{D0}S\rho_R H_a}{2m}$$

$$A_2 = \frac{C_{LR}S\rho_R H_a}{2m}$$

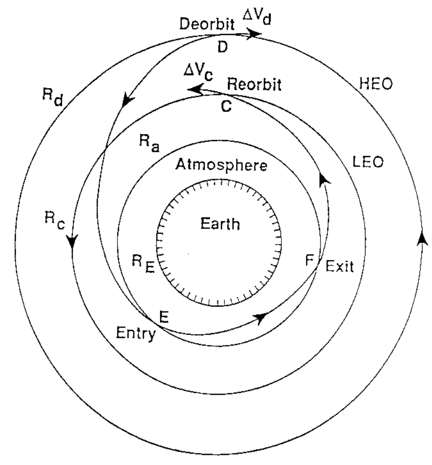

Fig. 1 Aeroassisted coplanar orbital transfer.

the normalized equations of motion become[5-7]

$$\frac{dh}{d\tau} = bv\sin\gamma \tag{1}$$

$$\frac{dv}{d\tau} = -A_1 b(1 + c^2)\rho_n v^2 - \frac{b^2\sin\gamma}{(b - 1 + h)^2} \tag{2}$$

$$\frac{d\gamma}{d\tau} = A_2 bc\rho_n v + \frac{bv\cos\gamma}{(b - 1 + h)} - \frac{b^2\cos\gamma}{(b - 1 + h)^2 v} \tag{3}$$

For the normalized system Eqs. (1–3), the state variables are $x = [h, v, \gamma]^T$ and the control variable is $u = c$.

Boundary Conditions

At the atmospheric entry ($\tau_0 = 0$) and exit ($\tau_f = T$) points certain conditions must be satisfied. The altitude must satisfy[5]

$$h(\tau_0) = 1, \qquad h(\tau_f) = 1 \tag{4}$$

The velocity and flight-path angle at entry and exit must satisfy

$$\left[2 - v^2(\tau_0)\right]a_d^2 - 2a_d + v^2(\tau_0)\cos^2\gamma(\tau_0) = 0 \tag{5}$$

$$\left[2 - v^2(\tau_f)\right]a_c^2 - 2a_c + v^2(\tau_f)\cos^2\gamma(\tau_f) = 0 \tag{6}$$

Optimization Problem

For minimum fuel consumption the performance index is given by

$$J = \Delta v = \Delta v_d + \Delta v_c = \phi[x(\tau_f), x(\tau_0)] \tag{7}$$

where

$$\Delta v_d = \sqrt{\frac{1}{a_d} - \frac{v(\tau_0)}{a_d}\cos[-\gamma(\tau_0)]}$$
$$\Delta v_c = \sqrt{\frac{1}{a_c} - \frac{v(\tau_f)}{a_c}\cos[\gamma(\tau_0)]} \tag{8}$$

Δv being the sum of characteristic velocities associated with Δv_d, the propulsive burn from HEO, and the final characteristic velocity associated with Δv_c, the propulsive burn into LEO.

The problem is to find the control variable c (normalized lift coefficient) that results in a minimum value of the performance index Eq. (7) subject to the dynamic constraint Eqs. (1–3) and boundary conditions (4–6). Incorporating the two terminal constraints, the augmented performance index becomes

$$\bar{J} = \Delta v_c + \Delta v_d + \mu_1\Psi_1 + \mu_2\Psi_2 = \bar{\phi}[x(\tau_f), x(\tau_0)] \tag{9}$$

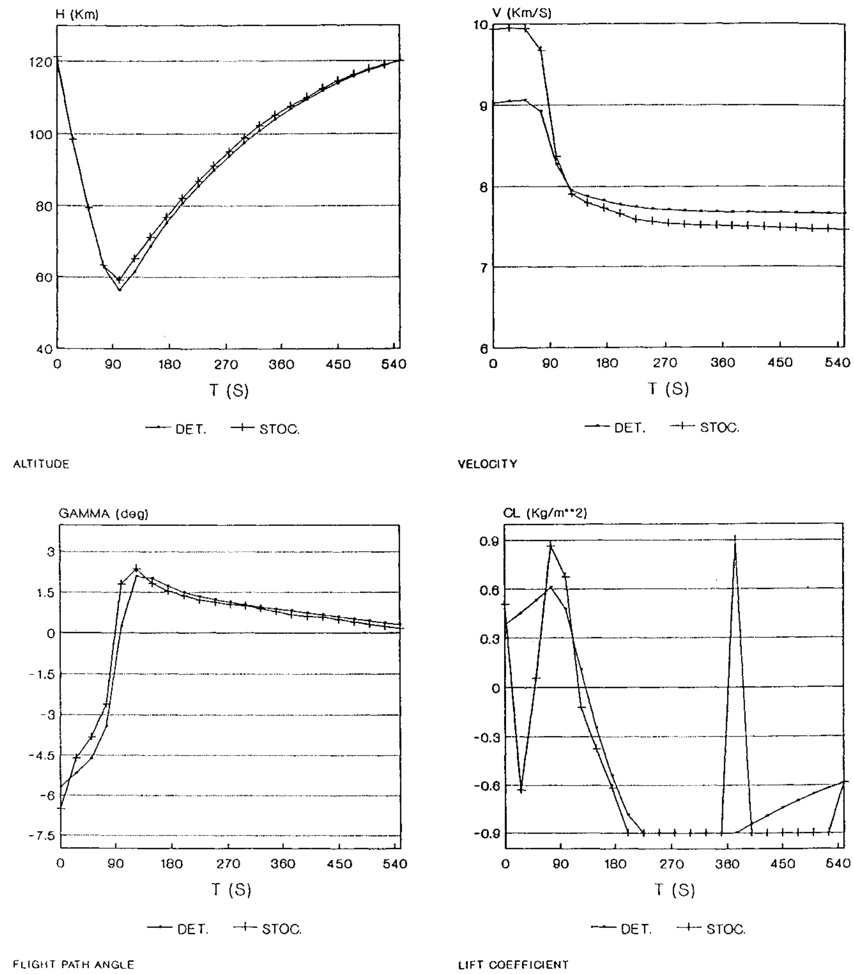

Fig. 2 Neighboring optimal state and control filter estimates (set #1).

where $\Psi_1 = h(\tau_f) - 1$ and Ψ_2 is given by Eq. (6). Using optimization principles,[2,6,11,13] the Hamiltonian function is given by

$$\mathcal{H} = P_h[bv \sin\gamma] + P_v\left[-A_1 b(1+c^2)\rho_n v^2 - \frac{b^2 \sin\gamma}{(b-1+h)^2}\right]$$

$$+ P_\gamma\left[A_2 bc\rho_n v + \frac{bv\cos\gamma}{b-1+h} - \frac{b^2\cos\gamma}{(b-1+h)^2 v}\right] \tag{10}$$

where P_h, P_v, P_γ are the costate variables. For the unconstrained problem, the optimal control is

$$\frac{\partial\mathcal{H}}{\partial c} = 0, \qquad c = \frac{A_2 P_\gamma}{2A_1 P_v v} \tag{11}$$

and for the constraint problem, the optimal control is

$$c = \begin{cases} -c_{\max} & \text{if} \quad c \geq c_{\max} \\ \dfrac{A_2 P_\gamma}{2A_1 P_v v} & \text{if} \quad |c| \leq c_{\max} \\ c_{\max} & \text{if} \quad c \leq -c_{\max} \end{cases} \tag{12}$$

The adjoint (costate) variables are obtained from

$$\frac{\mathrm{d}P_h}{\mathrm{d}\tau} = -\frac{\partial\mathcal{H}}{\partial h}, \qquad \frac{\mathrm{d}P_v}{\mathrm{d}\tau} = -\frac{\partial\mathcal{H}}{\partial v}, \qquad \frac{\mathrm{d}P_\gamma}{\mathrm{d}\tau} = -\frac{\partial\mathcal{H}}{\partial\gamma} \tag{13}$$

Assuming fixed terminal time we have the boundary condition

$$P^T(\tau_f) = \left[\frac{\partial\phi(x)}{\partial x} + \mu^T\frac{\partial\Psi}{\partial x}\right]\bigg|_{\tau=\tau_f} \tag{14}$$

Table 1 Data

Spacecraft data	Physical data	Normalized data
$m/s = 0.3 \times 10^7$ kg/m^2	$R_a = 0.6498 \times 10^7$ m	$H_R = 120.0 \times 10^3$ m
$C_{D0} = 0.21$	$R_d = 0.12996 \times 10^8$ m	$V_R = 7.832$ m/s
$K = 01.67$	$R_c = 0.6558 \times 10^7$ m	$t_R = 887.16$ s
$C_{LR} = 0.3546$	$M = 0.3986 \times 10^{15}$ m^2/s^2	$\rho_R = 0.3996 \times 10^{-2}$ kg/m^2
$C_{\max} = 2.538$	$r_e = 0.6378 \times 10^7$ m	$a_c = 01.009$
$C_{L,\max} = 0.9$	$H_a = 120.0 \times 10^3$ m	$a_d = 02.000$

Method of Solution and Results

The system of differential Eqs. (1–3) and (13) together with boundary conditions (4–6) form a TPBVP. The constants μ_1, μ_2 of Eq. (14) are found so that the two terminal constraints (4) and (6) are satisfied. The solution of the TPBVP is obtained using the OPTSOL code developed by Deutsche Forschungs and Versuchsanstalt fur Luft Ranmfahrt (DFVLR) at Obenpfaffenhofen, the former West Germany.[12,14]

For the data of Table 1, Fig. 2 shows the time histories of altitude, velocity, flight-path angle, and lift coefficient. The spacecraft enters and exists the atmosphere at an altitude of 120 km. The minimum altitude reached is 56.39 km. The vehicle enters the atmosphere with a velocity of 9029 m/s and leaves the atmosphere with a speed of 7665.6 m/s, thus giving a velocity reduction of 1363.4 m/s. The spacecraft enters the atmosphere with an inclination of −5.665 deg and exits with 0.30242 deg. The vehicle enters the atmosphere with lift coefficient of 0.4, gradually increases to 0.61, then gradually decreases to minimum lift coefficient, and then gradually increases during the remaining flight. The minimum fuel transfer requires a deorbit (impulse) characteristic velocity ΔV_d of 1045.65 m/s and a reorbit characteristic velocity $\Delta V_c = 200.83$ m/s, with a total characteristic velocity of 1246.47 m/s. When this aeroassisted transfer is compared with the Hohmann transfer, which is maneuvered entirely in outer space and has a total characteristic velocity of 2194 m/s, this shows that the saving due to coplanar aeroassisted transfer over the Hohmann transfer is 57%. The solution of the TPBVP is presented in Ref. 5.

III. Deterministic Neighboring Optimal Control Problem

The perturbation control problem considered in this section is in terms of small changes in initial and terminal conditions. Since the objective is to keep the actual state vector $x(\tau)$ near its ideal desired value $x_0(\tau)$ (in this case the optimal path computed in Sec. II), the actual plant input $u(\tau)$ must in fact be different from the precomputed ideal input $u_0(\tau)$. Expanding Eqs. (1–3), (5), and (6) about $x_0(\tau)$, $u_0(\tau)$ in a Taylor series up to first order, augmented performance index up to second order,[6,7,11] denoting state perturbation by $\delta x(\tau)$ and control perturbation by $\delta u(\tau)$ as $\delta x(\tau) = x(\tau) - x_o(\tau)$ and $\delta u(\tau) = u(\tau) - u_o(\tau)$, we obtain the following deterministic linear quadratic control problem:

Minimize

$$\delta^2 \bar{J} = \frac{1}{2}\left[\delta x^T\left(\phi_{xx} + \mu^T \Psi_{xx}\right)\delta x\right]\Big|_{\tau=\tau_f} + \frac{1}{2}\int_{\tau_0}^{\tau_f} \left[\delta x^T \delta u^T\right]\begin{bmatrix} \mathcal{H}_{xx}^{(1)} & 0 \\ 0 & \mathcal{H}_{uu}\end{bmatrix}\begin{bmatrix}\delta x \\ \delta u\end{bmatrix} \mathrm{d}\tau \tag{15}$$

Subject to constraints

$$\delta\dot{x}(\tau) = f_x^{(1)}(\tau)\,\delta x(\tau) + f_u(\tau)\,\delta u(\tau), \quad \delta x(\tau_0) \tag{16}$$

$$\delta\Psi = \Psi_x(\tau_f)\,\delta x(\tau_f) \tag{17}$$

where $\mathcal{H}_{xx}^{(1)} = \mathcal{H}_{xx} - \mathcal{H}_{xu}\,\mathcal{H}_{uu}^{(1)}\,\mathcal{H}_{ux}$, $f_x^{(1)} = f_x - f_u\,\mathcal{H}_{uu}^{-1}\,\mathcal{H}_{ux}$. However, since there is a terminal constraint to be met [Eq. (17)], the importance of controllability of Eq. (16) should be addressed. Thus, it is critical to know whether there exists a control δu that drives the state of Eq. (16) from some initial value $\delta x(\tau_0)$ to $\delta x(\tau_f)$. The system given by Eq. (16) is found to be controllable by solving the following matrix differential equation[15]:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}W(\tau,\tau_f) = f_x^{(1)}(\tau)W(\tau,\tau_f) + W(\tau,\tau_f)f_x^{(1)^T}(\tau) - f_u(\tau)f_u^T(\tau), \qquad W(\tau_f,\tau_f) = 0 \tag{18}$$

where the solution $W(\tau_0, \tau_f)$ is nonnegative definite for $\tau \geq \tau_0$. The matrix W is the so-called controllability Grammian.[15]

Anticipating the nature of the LQG control problem, which states that the filtering (estimation) and control problem can be solved independently[16,17] (separation principle), the next step would be to determine the optimal-control law.[8] This is nothing more than determining the control law that minimizes the variational cost of Eq. (15) subject to complete observations given by the state constraints (16) and (17). The solution of this optimization problem is then given by the following feedback control law[6,7,11]:

$$\delta u(\tau) = -\mathcal{H}_{uu}^{-1}\left[f_u^T\left(S - RQ^{-1}R^T\right)\delta x(\tau) + f_u^T R Q^{-1}\delta\Psi(\tau_f)\right] = -\Lambda_1(\tau)x(\tau) - \Lambda_2(\tau)\Psi(\tau_f) \tag{19}$$

where S, R, Q satisfy the Riccati equations

$$\dot{S} = -Sf_x^{(1)} - f_x^{(1)^T}S + Sf_u\mathcal{H}_{uu}^{-1}f_u^T - \mathcal{H}_{xx}^{(1)} = -Sf_x - f_x^T S + (Sf_u + \mathcal{H}_{xu})\mathcal{H}_{uu}^{-1}\left(\mathcal{H}_{ux} + f_u^T S\right) - \mathcal{H}_{xx}, \quad S(\tau_f) = \left[\phi_{xx} + \mu^T\Psi_{xx}\right]\Big|_{\tau=\tau_f} \tag{20}$$

$$\dot{R} = -\left(f_x^{(1)^T} - Sf_u\mathcal{H}_{uu}^{-1}f_u^T\right)R, \qquad R(\tau_f) = \Psi_x^T\Big|_{\tau=\tau_f} \tag{21}$$

$$\dot{Q} = R^T f_u - \mathcal{H}_{uu}^{-1} f_u^T, \qquad Q(\tau_f) = 0 \tag{22}$$

With $x = (h, v, \gamma)$, $u = c$, Eq. (19) becomes the deterministic neighboring control law of Sec. II. Moreover, solving Eqs. (20–22), one can obtain the control gains $\Lambda_1(\tau)$, $\Lambda_2(\tau)$.

IV. Stochastic Neighboring Optimal Control Problem

Stochastic Modeling

Recall that for the deterministic model [Sec. II, Eqs. (1–3)] certain assumptions were made.[4] Adding white-noise processes, the normalized equations of motion are represented by the following stochastic differential equations:

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = bv\,\sin\gamma + w_{a1} \tag{23a}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = A_1 b(1+c^2)\rho_n v^2 - \frac{b^2\sin\gamma}{(b-1+h)^2} + w_{a2} \tag{23b}$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = A_2 bc\rho_n v - \frac{bv\cos\gamma}{b-1+h} - \frac{b^2\cos\gamma}{(b-1+h)^2 v} + w_{a3} \tag{23c}$$

Likewise, adding white-noise processes to the signal components and assuming there are sensors that can measure altitude and altitude rate, we write

$$z_1 = h + n_{a1}, \qquad z_2 = v\sin\gamma + n_{a2} \tag{24}$$

The white-noise components w_{a1}, w_{a2}, and w_{a3} are used to account for neglected modeling dynamics, oblateness of Earth, incomplete knowledge of the aerodynamic forces, terms arising due to coriolis force, unknown parameters modeled only approximately (i.e., atmospheric density dependence on altitude by an exponential function, parabolic drag polar, etc.), and additional forces such as winds acting upon the space vehicle. The white-noise components n_{a1} and n_{a2} are used to model measurement errors and uncertainties, resulting in the classical "signal in additive noise" model.

Statistical Description

Recall that in the deterministic case the initial state of the system is supposed to be known exactly; however, this assumption no longer holds because in the stochastic environment, the initial state vector cannot be measured exactly. The initial state vector is assumed to be a vector-valued Gaussian random variable. Its initial mean and covariance matrix represent a priori information about the initial condition of the plant.

The uncertainty in the overall physical system is modeled in three parts; initial uncertainty in the state variable, plant uncertainty, and measurement uncertainty. The initial state vector is Gaussian with known mean and covariance matrix;

$$E[h_i\; v_i\; \gamma_i]^T = [\bar{h}_o\; \bar{v}_o\; \bar{\gamma}_o]^T \qquad \text{(assumed known)} \tag{25a}$$

$$E[(h_i - \bar{h}_o)(v_i - \bar{v}_o)(\gamma_i - \bar{\gamma}_o)]^T \times [(h_i - \bar{h}_o)(v_i - \bar{v}_o)(\gamma_i - \bar{\gamma}_o)] = \bar{P}_o \tag{25b}$$

where $\bar{P}_0 = \bar{P}_0^T \geq 0$ and assumed to be known. The notation h_i, v_i, γ_i represents the values of h, v, γ at $\tau = \tau_0$. The plant noise processes are defined by $w_{a1} \triangleq l_i w_i$ for $i = 1, 2, 3$, with white noise w_i having unit variance. Thus,

$$E[w_{a1}(\tau)\; w_{a2}(\tau)\; w_{a3}(\tau)]^T = [0 \quad 0 \quad 0]^T \qquad \text{for all} \quad \tau \geq \tau_0 \tag{26a}$$

$$E[w_{a1}(t)\; w_{a2}(t)\; w_{a3}(t)]^T[w_{a1}(\tau)\; w_{a2}(\tau)\; w_{a3}(t)] = L(t)\delta(t-\tau) \tag{26b}$$

where $L(\tau) > 0$ for all $\tau \geq \tau_0$ and assumed to be known. The measurement driving noises are defined by $n_{ai} \triangleq \theta_i n_i$ for $i = 1, 2$, with white noise n_i having zero mean and unit variance:

$$E[n_{a1}(\tau)\, n_{a2}(\tau)]^T = [0 \quad 0]^T, \qquad \forall\tau \geq \tau_0$$
$$E[n_{a1}(t)\, n_{a2}(t)]^T[n_{a1}(\tau)\, n_{a2}(\tau)] = \theta(t)\delta(t-\tau) \tag{27}$$

where $\theta(\tau) > 0$ for all $\tau \geq \tau_0$ and assumed to be known. Furthermore, $w(\tau)$, $n(\tau)$, $x(\tau_0)$ are assumed to be mutually independent.

Stochastic Linear Quadratic Control Problem

Following the same approach as in Sec. III, using a Taylor series expansion about $x_o(\tau) = [h_o(\tau)\, v_o(\tau)\, \gamma_o(\tau)]^T$, $u_o(\tau) = c_o(\tau)$,

$$\begin{bmatrix} \delta\dot{h}(\tau) \\ \delta\dot{v}(\tau) \\ \delta\dot{\gamma}(\tau) \end{bmatrix} = f_x(\tau) \begin{bmatrix} \delta h(\tau) \\ \delta v(\tau) \\ \delta\gamma(\tau) \end{bmatrix} + f_u(\tau)\,\delta c(\tau) + \begin{bmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{bmatrix} \begin{bmatrix} w_1(\tau) \\ w_2(\tau) \\ w_3(\tau) \end{bmatrix} \tag{28}$$

$$\begin{bmatrix} \delta z_1(\tau) \\ \delta z_2(\tau) \end{bmatrix} = g_x(\tau) \begin{bmatrix} \delta h(\tau) \\ \delta v(\tau) \\ \delta\gamma(\tau) \end{bmatrix} + \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix} \begin{bmatrix} n_1(\tau) \\ n_2(\tau) \end{bmatrix} \tag{29}$$

The performance index differs from the one given by Eq. (15) in that $\delta^2 \bar{J}$ is random rather than deterministic because δh, δv, $\delta\gamma$, and δc are now random processes and the terminal cost is augmented by the quantity $\delta x^T(\tau_f)\Psi_x^T(\tau_f)N\Psi_x(\tau_f)\,\delta x(\tau_f)$. Thus,

$$\delta^2 \bar{J}^0 \triangleq \tfrac{1}{2}E\left\{ \left[\delta x^T\left(\phi_{xx} + \Psi_x^T N \Psi_x + \mu^T \Psi_{xx}\right)\delta x\right]\Big|_{\tau=\tau_f} + \int_{\tau_0}^{\tau_f} [\delta x^T\ \delta u^T] \begin{bmatrix} \mathcal{H}_{xx} & \mathcal{H}_{xu} \\ \mathcal{H}_{ux} & \mathcal{H}_{uu} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} \mathrm{d}\tau \right\} \tag{30}$$

The reason for augmenting the performance index follows from the fact that in the presence of random inputs the terminal constraint $\Psi[x(\tau_f)] = 0$ cannot be satisfied exactly.[11] Thus in place of the linear terminal constraint $\delta\Psi = \Psi_x\,[x(\tau_f)]\delta x(\tau_f) = 0$ additional terminal penalty cost as shown in Eq. (30) is included to ensure that $\Psi_x[x(\tau_f)]\,\delta x(\tau_f)$ stays near zero as τ approaches τ_f when the current actions of $\delta u(\tau)$ are not as strongly felt since they take time to excite the system. The matrix N is positive definite and chosen so that $E\{\mathrm{diag}(\delta\Psi\,\delta\Psi^T)\}$ when evaluated at the final time have acceptable values. This process requires the evaluation of $\Psi_x E(\delta x\,\delta x^T)\,\Psi_x^T$. This is the same as evaluating $\Psi_x X \Psi_x^T$ where $X = E[\delta x\,\delta x^T]$ is the mean-square value of $\delta x(\tau)$ and satisfies

$$\dot{X}_r = [f_x(\tau) - f_u\Lambda(\tau)]X(\tau) + X(\tau)[f_x(\tau) - f_u(\tau)\Lambda(\tau)]^T + L(\tau), \qquad X(\tau_0) = X_0 \tag{31}$$

with $\Lambda(\tau)$ the control gain to be specified later.

Solution of Linear Quadratic Gaussian Control Problem

The problem is to find the control law $\delta c(\tau)$ such that, given the linearized dynamic system Eq. (28) and the linearized observation Eq. (29) whose measurements are $\delta z(\sigma)$, $\tau_0 \le \sigma \le \tau_f$, the cost given by Eq. (30) is minimized. Since in the linear case the control depends on the measurements only and separation therefore holds,[16,17] the conditional mean satisfies the following linear stochastic differential equation (Kalman-Bucy filter)[9]:

$$\begin{bmatrix} \delta\dot{\hat{h}}(\tau) \\ \delta\dot{\hat{v}}(\tau) \\ \delta\dot{\hat{\gamma}}(\tau) \end{bmatrix} = f_x(\tau) \begin{bmatrix} \delta\hat{h}(\tau) \\ \delta\hat{v}(\tau) \\ \delta\hat{\gamma}(\tau) \end{bmatrix} + f_c(\tau)\,\delta c(\tau) + K_f(\tau)\left\{ \begin{bmatrix} \delta z_1(\tau) \\ \delta z_2(\tau) \end{bmatrix} - g_x(\tau) \begin{bmatrix} \delta\hat{h}(\tau) \\ \delta\hat{v}(\tau) \\ \delta\hat{\gamma}(\tau) \end{bmatrix} \right\} \tag{32}$$

$$\begin{bmatrix} \delta\hat{h}(\tau_0) \\ \delta\hat{v}(\tau_0) \\ \delta\hat{\gamma}(\tau_0) \end{bmatrix} = \begin{bmatrix} \bar{h}_o - h_o(\tau_0) \\ \bar{v}_o - v_o(\tau_0) \\ \bar{\gamma}_o - \gamma_o(\tau_0) \end{bmatrix}, \qquad \delta c(\tau) = -\Lambda(\tau) \begin{bmatrix} \delta\hat{h}(\tau) \\ \delta\hat{v}(\tau) \\ \delta\hat{\gamma}(\tau) \end{bmatrix}$$

with the control gains $\Lambda(\tau)$ and filter gains $K_f(\tau)$ still to be specified.

Control and Filter Gains

1) *Control gains:* Using the LQG control property, which states that the optimal-control correction $\delta u(\tau)$ is generated from the estimated conditional mean $\delta\hat{x}(\tau)$, $\delta u(\tau)$ is given by the relationship

$$\delta u(\tau) = -\Lambda(\tau)\,\delta\hat{x}(\tau) \tag{33}$$

where the gain matrix $\Lambda(\tau)$ is the one determined from the deterministic linear quadratic problem. This can be shown as follows. Recall the deterministic solution (see Sec. III)

$$\delta u(\tau) = -\Lambda_1(\tau)\,\delta x(\tau) - \Lambda_2(\tau)\,\delta\Psi(\tau_f) \tag{34}$$

Since the deterministic system Eq. (16) is controllable and there is an additional terminal cost added in Eq. (30) to penalize any deviations of $\delta\Psi x(\tau_f)$ from zero, the assumption that $\delta\Psi x(\tau_f) = 0$ leads to

$$\delta u(\tau) = -\Lambda_1(\tau)\,\delta x(\tau) \tag{35}$$

However, since $\delta u(\tau)$ is not deterministic but instead is a random process, $\delta x(\tau)$ of Eq. (35) should be replaced by $\delta\hat{x}(\tau)$. Therefore, the control gain is given by

$$\delta u(\tau) = -\mathcal{H}_{uu}^{-1}\left[\mathcal{H}_{ux} + f_u^T(S - RQ^{-1}R^T)\right]\delta\hat{x}(\tau) \tag{36}$$

where S, Q, R satisfy matrix equations (20–22) with the terminal condition $S(\tau_f)$ augmented by $\Psi_x^T N \Psi_x$ [see Eq. (30)].

2) *Filter gains:* The optimal estimation gain is determined from

$$K_f(\tau) = P(\tau)g_x^T(\tau)\theta^{-1}(\tau) \tag{37}$$

where the covariance matrix $P(\tau)$ is found by solving the matrix differential equation

$$\dot{P}(\tau) = f_x^{(1)}(\tau)P(\tau) + P(\tau)f_x^{(1)T}(\tau) - P(\tau)g_x^T(\tau)\theta^{-1}(\tau)g_x(\tau)P(\tau) + L(\tau), \qquad P(\tau_0) = \bar{P}_0 \tag{38}$$

Therefore, it follows that the true state and control trajectories when analyzed in a stochastic environment are given by

$$\begin{bmatrix} \hat{h}(\tau) \\ \hat{v}(\tau) \\ \hat{\gamma}(\tau) \end{bmatrix} = \begin{bmatrix} h_0(\tau) \\ v_0(\tau) \\ \gamma_0(\tau) \end{bmatrix} + \begin{bmatrix} \delta\hat{h}(\tau) \\ \delta\hat{v}(\tau) \\ \delta\hat{\gamma}(\tau) \end{bmatrix}, \qquad c(\tau) = c_0(\tau) + \delta c(\tau) \tag{39}$$

where $\delta c(\tau)$ is given by Eq. (36).

Performance Index Evaluation

The average cost of the perturbed stochastic control problem given by Eq. (30) can be expressed[11] as

$$\delta^2 \bar{J}^0 = \tfrac{1}{2}\,\mathrm{Tr}\left\{ S(\tau_0)X(\tau_0) + \int_{\tau_0}^{\tau_f} \left[S(t)L(t) + \Lambda^T(t)\mathcal{H}_{cc}(t)\Lambda(t)P(t)\right]\mathrm{d}t \right\} \tag{40}$$

where Tr indicates the trace operator. Finally, following Ref. 11 and assuming that the combined optimization of the nominal trajectory and the LQG problem is given by

$$\min E\left(J_{\mathrm{nom}} + \delta^2\bar{J}\right) = \min[J_{\mathrm{nom}}]\big|_{\mathrm{noise}=0} + \min\left[\delta^2\bar{J}_{\mathrm{av}}\right] \tag{41}$$

the total minimum cost can be evaluated.

Numerical Algorithms

The implementation of the Kalman–Bucy filter [i.e., Eq. (32)] requires additional calculations as follows. Initially the deterministic optimal solution of Sec. II was stored in the memory of a computer. Since the above solution was given only at the grid points or nodes (as required by the OPTSOL code), a cubic spline interpolation method is utilized that is implemented as a member of the IMSL library. This requires IMSL routines DCSINT and DCSDER, the first to obtain the cubic spline coefficients and the second to evaluate the optimal trajectory at a particular time. Then, the controllability Grammian matrix $W(\tau, \tau_f)$ [i.e., Eq. (18)] is solved using the Gear backward difference method with an acceptable local error of order 10^{-12} (in terms of normalized data), again through the routine DIVPAG of the IMSL Library. Next, the control gain was obtained by solving Eqs. (20–22) using the same algorithm as above. Since the Kalman–Bucy filter is driven by the measurements $\delta z(\tau)$, a computer simulation is performed to generate noisy measurements $\delta z(\tau)$. This was accomplished by solving the stochastic Eq. (28) for $\delta h(\tau)$, $\delta v(\tau)$, $\delta\gamma(\tau)$ for $\tau_0 \leq \tau \leq \tau_f$ and then generating the noisy measurements through Eq. (29). The state and measurement noise components are chosen to have zero mean and unit variance. Then, the matrix solutions of the mean-square value of the state [i.e., Eq. (31)] and covariance matrix $P(\tau)$ [i.e., Eq. (38)] were obtained. However, the implementation of the both equations requires special attention since their solution should be positive definite, a property violated through usual integration techniques. This is due to roundoff errors or iterates leaving the cone of positive-definite matrices, which is caused by systems that are on the borderline or controllability, a situation that resulted by solving Eq. (18).[10] Thus, a square-root algorithm was implemented that was presented in Ref. 18 serving two purposes. First, being a square-root technique, it entails all the advantages of the square-root algorithm, such as nonnegative definiteness and accuracy. Second, it provides the eigenvalues and eigenvectors of the solution matrix continuously and then decomposes the solution matrix into eigenvalues and eigenvectors. The matrix Riccati equations were solved utilizing Runge–Kutta–Verner fifth- and sixth-order integration methods, which are again implemented using the IMSL Library routine. The initial step of the integration was set to be $\Delta\tau = 10^{-10}$ (normalized time). The integration routine automatically divides the initial step size to meet a given error measure of 10^{-10}. Finally, the Kalman–Bucy filter [i.e., Eq. (32)] is integrated using the previous algorithm and the conditional mean vector $\delta\hat{x}(\tau)$ is obtained. However, several implementations are required to choose an acceptable matrix N [see Eq. (30)] to force $\delta\Psi[x(\tau_f)]$ near zero.

V. Numerical Data and Results of Stochastic Control Problem

The data used for simulations are borrowed from Ref. 3, and are given in Table 1 as spacecraft and physical data. Some of the normalized data are also given in the same table. The simulations performed to obtain the neighboring altitude, velocity, flight-path angle, and control estimates shown in Figs. 2–4 required additional initial conditions and noise variances, which are summarized Table 2. The diagonal matrix N and cost computed through Eq. (40) are also given.

Most of the difficulties that are encountered with the Kalman-Bucy filter implementation [i.e., Eq. (32)] are primarily related to choosing the diagonal matrix N, the white-noise intensities l_1, l_2, l_3, θ_1, θ_2 and the initial error covariance matrix $\bar{P}_0$. As shown in Sec. IV the diagonal matrix N is chosen so that the mean-square value of the state [i.e., Eq. (31)] was acceptable with values near $\tau = \tau_f$. Using this criterion, the diagonal matrix N (normalized) is chosen as shown in Table 2. The noise intensities are obtained after several simulations to keep the filter from diverging and at the same time to prevent the value of the cost $\delta^2\bar{J}^0$ from increasing beyond the nominal cost, since these intensities have a direct effect on the cost of Eq. (30).

Figure 2 shows that the actual trajectories of altitude and flight-path angle follow the nominal trajectories closely. The initial altitude is 1200 m above its nominal value and reduces to 143.9 m at the final time. The initial flight-path angle is −0.85 deg below its nominal value and reduces to −0.147 deg at the final time. The velocity, when started at 902.9 m/s above its nominal value, reduces to 200.516 m/s

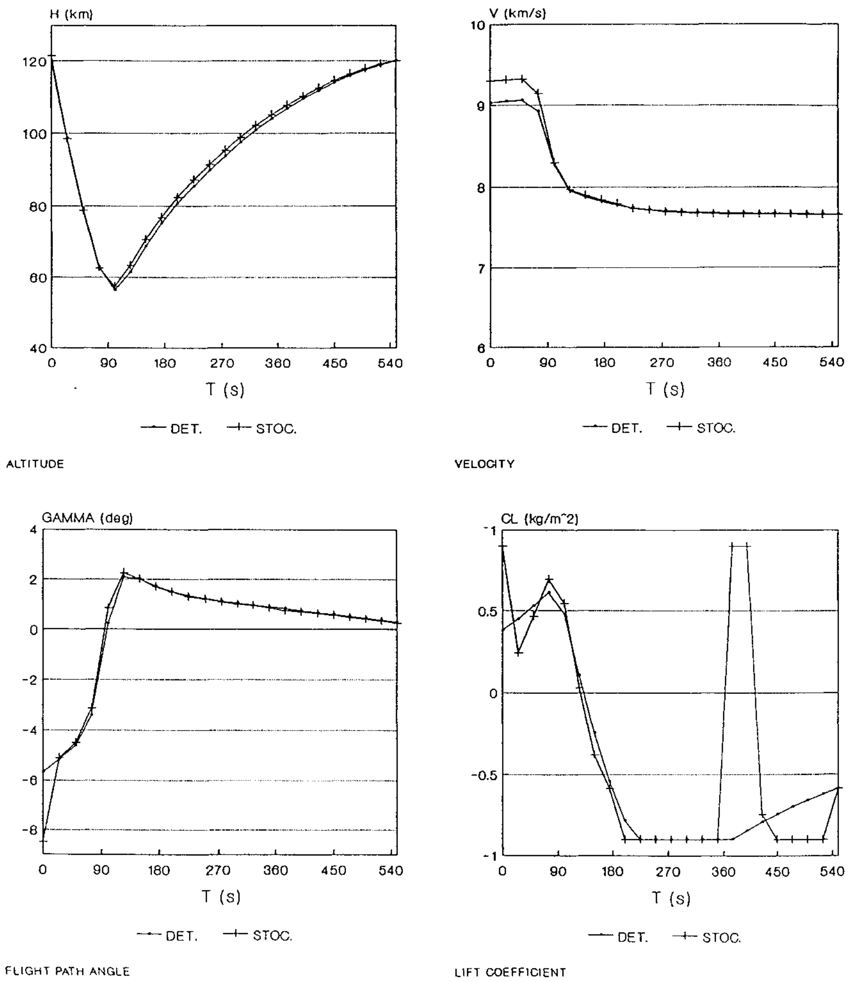

Fig. 3 Neighboring optimal state and control filter estimates (set #2).

Table 2 Data

Data	Set 1			Set 2			Set 3		
l_1		5.0×10^{-3}			5.0×10^{-3}			5.0×10^{-3}	
l_2		1.0×10^{-2}			1.0×10^{-2}			1.0×10^{-3}	
l_2		1.0×10^{-2}			1.0×10^{-2}			1.0×10^{-2}	
θ_1		1.0×10^{-3}			1.0×10^{-3}			1.0×10^{-3}	
θ_2		1.0×10^{-3}			1.0×10^{-3}			1.0×10^{-3}	
	2.1×10^{-4}	0	0	2.1×10^{-4}	0	0	2.1×10^{-4}	0	0
X_0	0	12.3×10^{-4}	0	0	12.3×10^{-4}	0	0	12.3×10^{-4}	0
	0	0	28.7×10^{-4}	0	0	28.7×10^{-4}	0	0	28.7×10^{-4}
	1.0×10^{-4}	0	0	1.0×10^{-4}	0	0	1.0×10^{-4}	0	0
P_0	0	3.0×10^{-4}	0	0	3.0×10^{-4}	0	0	3.0×10^{-4}	0
	0	0	7.0×10^{-4}	0	0	7.0×10^{-4}	0	0	7.0×10^{-4}
N	0.1		0	0.1		0	0.1		0
	0		0.1	0		0.1	0		0.1
$\delta\hat{h}(\tau_0)$		1200 m			1440 m			1200 m	
$\delta\hat{v}(\tau_0)$		902.9 m/s			270 m/s			902.9 m/s	
$\delta\hat{\gamma}(\tau_0)$		−0.850 deg			−2.832 deg			−1.33 deg	
$\delta^2 \bar{J}$ deg		608.755 m/s			1143.86 m/s			609.79 m/s	

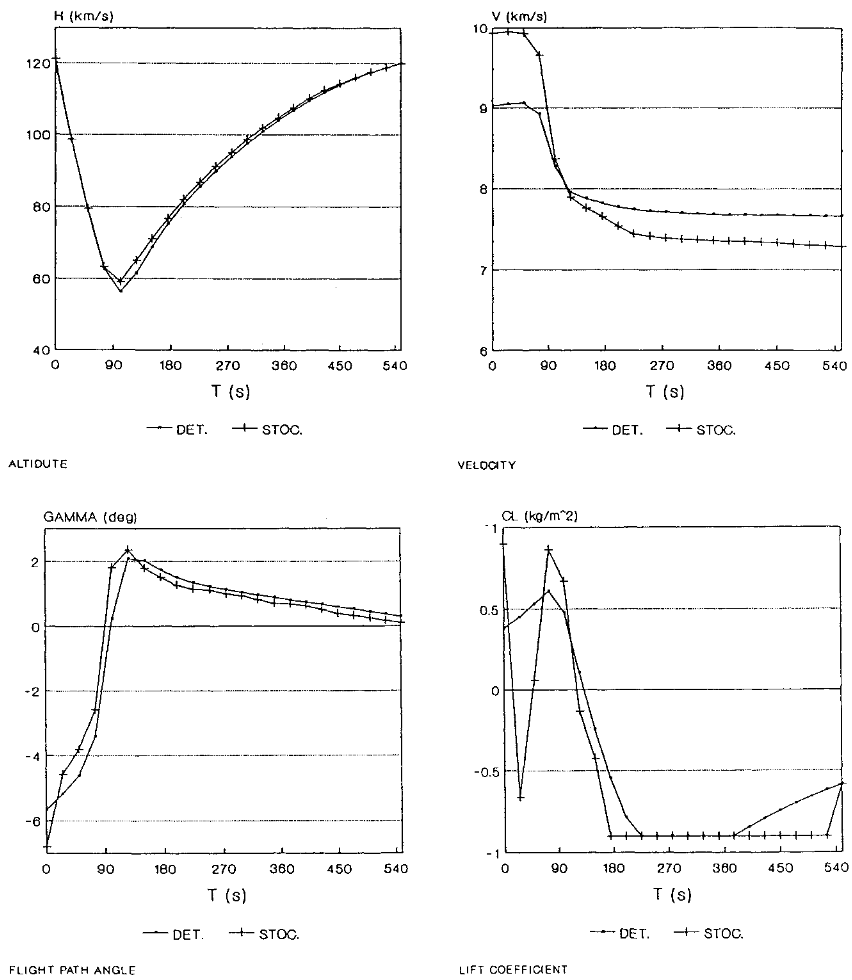

Fig. 4 Neighboring optimal state and control filter estimates (set #3).

below its nominal value. The lift coefficient is shown to behave in a random fashion without following its nominal path. This is as expected since the control moves toward the direction that best reduces the error to zero.

Figure 3 shows that all three trajectories are kept near their nominal path. The initial values of altitude, velocity, and flight-path angle are 1440.0 m, 270.0 m/s, and −2.832 deg, respectively. The final deviations of altitude, velocity, and flight-path angle are 42 m, −7.81 m/s, and −0.0632 deg, respectively. The lift coefficient starts at its maximum allowable value, which results in a reduction of velocity as in the deterministic case.

Figure 4 shows the same behavior as Fig. 3. That is, tracking of altitude and flight-path angle is achieved. However, the velocity deviates from the nominal trajectory by −376.39 m/s at the final time, where an initial deviation of 903 m/s is used. The initial deviation of altitude and flight-path angle are 1200 m/s and −1.33 deg, respectively. The deviations of altitude and flight-path angle are 133.84 m and −0.193 deg, respectively.

VI. Conclusions

The objective of this study was to analyze the neighboring optimal guidance of the AOTV problem using the well-known Kalman filter to incorporate modeling and measurement uncertainties. However, before the optimal filter estimates are computed, the optimal solution of the TPBVP as well as the control gain of the neighboring optimal-control problem (linear quadratic) are required and are computed off-line.

The simulations show that the filter estimates obtained from the solution of the LQG problem behave relatively well in keeping the actual state trajectory near its ideal (reference) state trajectory, at the expense of increasing the total cost. In fact, the main contribution of this increased cost is due to the requirement that the final state should lie on a certain terminal manifold.

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